TOUCHING CONICS GENERATED BY FEUERBACH CONFIGURATIONS AND HOMOTHETY

Sava Grozdev, Veselin Nenkov, Tatyana Madjarova

Abstract. *The following assertion gives an interesting relation between the circumcircle of a triangle and the circle which is inscribed in one of its angles: A circle ω with centre I touches the sides AB and AC of $\triangle ABC$ intersecting the side BC in points X and Y. If Z is the common point of the line through A parallel to BC and the line through I perpendicular to BC, then the circumcircles of $\triangle ABC$ and $\triangle XYZ$ are tangent. The present paper is dedicated to a generalization of that assertion concerning suitably constructed inscribed conics in an angle of a given triangle and a kind of its circumscribed curves.*

Key words: Triangle, Conic, Common Points.

Generalization of a geometric problem

Many geometric relations connected with circles could be generalized for suitable conics. Such a relation will be considered in the sequel generalizing the following problem: *The circle ω with center I touches the sides AB and AC of $\triangle ABC$ intersecting the side BC and X of Y. Let Z be the common point of the line through A parallel to BC and the line through I perpendicular to BC. Prove that the circumcircles of $\triangle ABC$ and $\triangle XYZ$ are tangent.*

It is clear that it is not possible to generalize the problem for arbitrary conics. Special ones are needed. A special type of conics unites the curves belonging to Feuerbach's configurations [2]. If I_0 is an arbitrary point in the plane of $\triangle ABC$ and the points I_A, I_B, I_C determine a harmonic triangle to I_0, then the midpoints of the segments I_0I_A, I_0I_B, I_0I_C, I_AI_B, I_BI_C and I_CI_A lie on a second degree curve $\bar{k}(O)$ with center O, which is the circumscribed one of $\triangle ABC$. The center O of $\bar{k}(O)$ defines conics in a unique way k, k_A, k_B and k_C with centers I_0, I_A, I_B and I_C, respectively. The curves k, k_A, k_B and k_C touch a special conic Ω, named Euler's curve and we say that the curves $\bar{k}(O)$, Ω, k, k_A, k_B and k_C determine a Feuerbach's configuration [2]. A basic property of these curves is that they are homothetic. Any two circles are homothetic [2, 3]. Concretely, the tangent circle to AB and AC from the formulated problem is homothetic to the incircle of $\triangle ABC$ with center of homothety A. This is the reason to consider the curve ω with center I, which touches the lines AB and AC being homothetic to k with center of homothety A and coefficient h. Denote by X and Y the intersection points of ω and BC. Let p be the line trough A parallel to BC, and q the line through I conjugated to BC with respect to $\bar{k}(O)$. The common point of p and q is denoted by Z.

For the generalization of the problem under consideration it is necessary to determine a suitable circumscribe curve ω_0 of $\triangle XYZ$. It turns out that the lines through the midpoints of the sides of $\triangle XYZ$ that are conjugate to the corresponding sides with respect to $\bar{k}(O)$ have a common point W. This is the reason to consider the circumscribe curve ω_0 of $\triangle XYZ$ with centre W. After several necessary constructions by means of the software program GSP (The Geometer's Sketchpad) we notice that the following assertion holds true:

Assertion 1. *The curves ω_0 and $\bar{k}(O)$ are tangent at T (Fig. 1, 2).*

In such a way we have formulated a generalization of the geometric problem under consideration.

Proof of Assertion 1

The formulation of Assertion 1 is motivated after a certain analysis of the problem under consideration and observations of the new configuration obtained by means of GSP. We need a strict proof. For the purpose we will use barycentric coordinates with coordinate triangle ABC, where $A(1,\,0,\,0)$, $B(0,\,1,\,0)$ and $C(0,\,0,\,1)$ [1].

Let the coordinate representation of the centre k be $I_0(x_I,\,y_I,\,z_I)$ $(x_I + y_I + z_I = 1)$. Then, for the centre ω we have that $I(h(x_I - 1) + 1,\, hy_I,\, hz_I)$. It is known from [2] and [3] that for the coordinates of the touching points B_I and C_I of k and the lines AC and AB respectively we have:

$$B_I\left(\frac{1-2z_I}{2y_I},\,0,\,\frac{1-2x_I}{2y_I}\right),\quad C_I\left(\frac{1-2y_I}{2z_I},\,\frac{1-2x_I}{2z_I},\,0\right) \tag{1}$$

It follows from (1) that

$$B_I'\left(\frac{2y_I - h(1-2x_I)}{2y_I},\, 0,\, \frac{h(1-2x_I)}{2y_I}\right),$$
$$C_I'\left(\frac{2z_I - h(1-2x_I)}{2z_I},\, \frac{h(1-2x_I)}{2z_I},\, 0\right), \tag{2}$$

where B_I' and C_I' are the tangent points of ω with the lines AC and AB respectively.

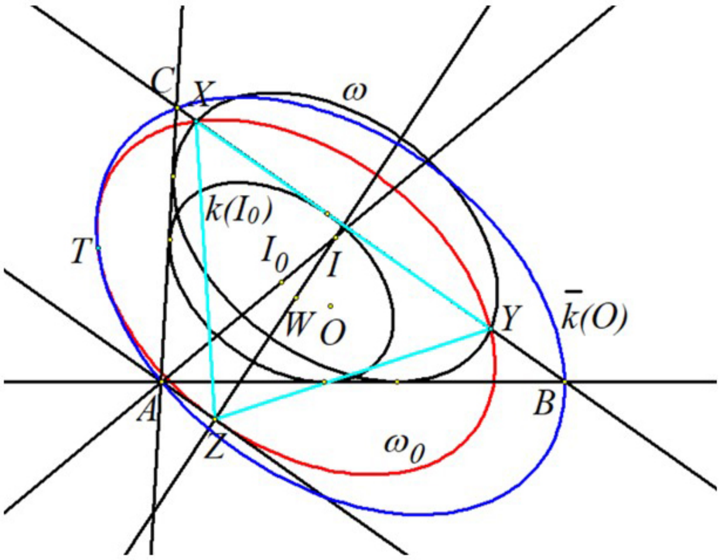

Figure 1.

As shown in [2], the equation of $\bar{k}(O)$ could be presented in the form

$$\bar{k}(O):\ x_I^2yz + y_I^2zx + z_I^2xy = 0. \tag{3}$$

Since the curve ω is homothetic to k, it is homothetic to $\bar{k}(O)$ too. One of the results in [4] implies the existence of a real number h_0 with which the presentation of the equation of ω takes the form:

$$\omega:\ h_0\left(x_I^2yz + y_I^2zx + z_I^2xy\right) + \left(a_{11}x + a_{22}y + a_{33}z\right)(x+y+z) = 0. \tag{4}$$

We have to determine the coefficients a_{11}, a_{22} and a_{33} depending on h_0. We will use the conditions for the points B_I' and C_I' to lie on ω, namely $h_0y_I^2zx + a_{11}x + a_{33}z = 0$ and $h_0z_I^2zx + a_{11}x + a_{22}y = 0$ (they follow from (2) and (4)). The conditions for the lines AC and AB to touch ω are the following $\left(a_{11}+a_{33}+h_0 y_I^2\right)-4a_{11}a_{33}=0$ and $\left(a_{11}+a_{22}+h_0 z_I^2\right)-4a_{11}a_{22}=0$ (they follow from (4)). Using the last four equalities we derive a_{22} and a_{33} by the use of a_{11}:

$$a_{22}=\frac{[h(1-2x_I)-2z_I]\,[h_0 h z_I(1-2x_I)+2a_{11}]}{2h(1-2x_I)},$$
$$a_{33}=\frac{[h(1-2x_I)-2y_I]\,[h_0 h y_I(1-2x_I)+2a_{11}]}{2h(1-2x_I)}.$$

The last two equalities and the previous ones imply that

$$a_{11}=\frac{1}{4}h_0 h^2\left(1-2x_I\right)^2,$$
$$a_{22}=-\frac{1}{4}\left[2z_I-h\left(1-2x_I\right)\right]^2,$$
$$a_{33}=-\frac{1}{4}\left[2y_I-h\left(1-2x_I\right)\right]^2.$$

For the equation of ω we obtain that

$$\omega:\ \begin{array}{l}4\left(x_I^2yz+y_I^2zx+z_I^2xy\right)-h^2\left(1-2x_I\right)^2x-\\ -\left[2z_I-h\left(1-2x_I\right)\right]^2y-\left[2y_I-h\left(1-2x_I\right)\right]^2z=0.\end{array}\tag{5}$$

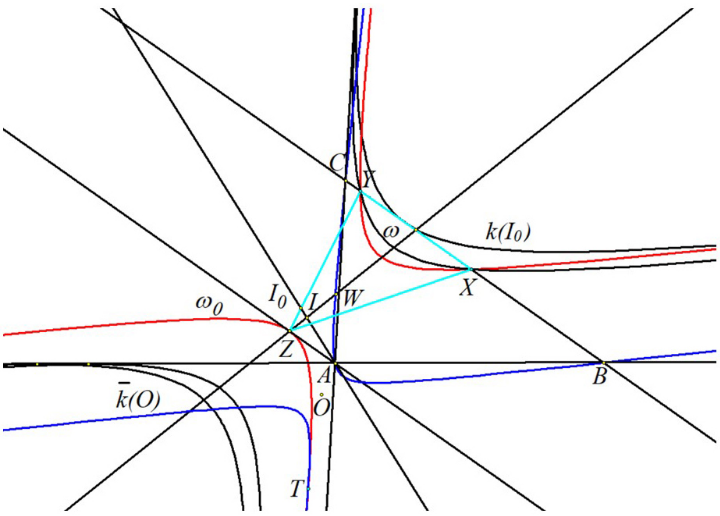

Figure 2.

The next step is to determine relations between the coordinates of the common points $X(0,\xi,1-\xi)$ and $X(0,\eta,1-\eta)$ of the curve ω and the line BC. For the purpose we replace the coordinates of the points X and Y in the equation (5) of ω with $x = 0$ and $z = 1 - y$. Thus, we obtain the quadratic equation:

$$4h^2x_I^2y^2 - 4\left(x_I^2 + y_I^2 - z_I^2 + 2hx_Iy_I - 2hz_Ix_I - hy_I + hz_I\right)y+ \\ + (2hx + 2y - h)^2 = 0$$

If ξ and η are the roots, then according to the Vietta formulae we have:

$$\begin{aligned} \xi + \eta &= \frac{\left(x_I^2 + y_I^2 - z_I^2 + 2hx_Iy_I - 2hz_Ix_I - hy_I + hz_I\right)}{h^2x_I^2}, \\ \xi\eta &= \frac{(2hx + 2y - h)^2}{4h^2x_I^2} \end{aligned} \tag{6}$$

For the coordinates of the point Z we use that it lies on the line $p : x = 1$ and the line q, determined by the midpoint M_0 and I. Here are the parametric equations of q:

$$q : \begin{cases} x = 1 - h(1 - x_I) + \left[1 - h\left(1 - x_I\right)\right]t, \\ y = hy_I + \left[2hy_I - \left(\xi + \eta\right)\right]t, \\ z = hz_I + \left[2hz_I - 2 - \left(\xi + \eta\right)\right]t. \end{cases}$$

The equations of p and $q \equiv IM_0$ together with the equalities (6) determine the coordinates of the point Z in the form:

$$Z = \left(1, \frac{h(1 - 2x_I)(z_I - y_I)}{2x_I^2}, \frac{h(1 - 2x_I)(y_I - z_I)}{2x_I^2}\right).$$

Finding the coordinates of the point W we notice, that the vectors

$$\begin{aligned} \vec{u_1} = \Big(u_{11} = 2x_I^2,\ u_{21} &= h(1 - 2x_I)(z_I - y_I) - 2x_I^2\xi, \\ u_{31} &= h(1 - 2x_I)(y_I - z_I) - 2x_I^2(1 - \xi)\Big), \\ \vec{u_2} = \Big(u_{12} = 2x_I^2,\ u_{22} &= h(1 - 2x_I)(z_I - y_I) - 2x_I^2\eta, \\ u_{32} &= h(1 - 2x_I)(y_I - z_I) - 2x_I^2(1 - \eta)\Big) \end{aligned}$$

are colinear with the lines XZ and YZ, respectively.

As shown in [5], the coordinates of their conjugate vectors $\vec{v_1}$ (v_{11}, v_{21}, v_{31}) and $\vec{v_2}$ (v_{12}, v_{22}, v_{32}) are expressed with the equalities:

$$\begin{aligned} v_{2j} &= y_I^2 u_{1j} + (x_I - z_I)(1 - y_I)u_{2j} - y_I^2 u_{3j}, \\ v_{3j} &= -z_I^2 u_{1j} + z_I^2 u_{2j} + (y_I - x_I)(1 - z_I)u_{3j}, \quad \text{where } j = 1,\, 2. \end{aligned}$$

The coordinates of the midpoints M_1 and M_2 of the segments XZ and YZ, respectively are expressed in the following way:

$$\begin{aligned} M_1 = \Big(m_{11} &= \frac{1}{2},\, m_{21} = \frac{h(z_I - y_I)(1 - 2x_I)}{4x_I^2} + \frac{\xi}{2}, \\ m_{31} &= \frac{h(y_I - z_I)(1 - 2x_I)}{4x_I^2} + \frac{1 - \xi}{2}\Big), \\ M_2 = \Big(m_{12} &= \frac{1}{2},\, m_{22} = \frac{h(z_I - y_I)(1 - 2x_I)}{4x_I^2} + \frac{\eta}{2}, \\ m_{32} &= \frac{h(y_I - z_I)(1 - 2x_I)}{4x_I^2} + \frac{1 - \eta}{2}\Big). \end{aligned}$$

The parametric equations of the lines s_1 and s_2, passing through the midpoints of XZ and YZ, also conjugated to XZ YZ, are:

$$s_1 : \begin{cases} x = m_{11} + v_{11}t_1, \\ y = m_{21} + v_{21}t_1, \\ z = m_{31} + v_{31}t_1, \end{cases} \qquad s_2 : \begin{cases} x = m_{12} + v_{12}t_2, \\ y = m_{22} + v_{22}t_2, \\ z = m_{32} + v_{32}t_2, \end{cases}$$

From here we obtain the coordinates x_W and y_W of $W(x_W,\, y_W,\, z_W)$:

$$\begin{aligned} x_W &= \frac{\vartheta_x}{2(1 - 2x_I)(1 - 2y_I)(1 - 2z_I)}, \\ y_W &= \frac{\vartheta_y}{4x_I^2(1 - 2x_I)(1 - 2y_I)(1 - 2z_I)}, \end{aligned}$$

where

$$\begin{aligned} \vartheta_x = \Big\{ & 2x_I^2\big[(z_I - y_I)(1 - 2x_I)h + 1 - 2z_I - 2x_I y_I\big](\xi + \eta) - \\ & - 4x_I^2 \xi\eta - (1 - 2x_I)^2 (y_I - z_I)^2 h^2 + \\ & + 2(1 - 2x_I)(y_I - z_I)(1 - 2z_I - 2x_I y_I)h - 4x_I^2 y_I^2 \Big\}, \end{aligned}$$

$$\vartheta_y = -\Big\{2x_I^2\big[(1-2x_I)(y_I-z_I)(1-2z_I-2x_Iy_I)h - 4x_I^2y_I^2\big](\xi+\eta)+$$
$$+\,4x_I^4(1-2z_I-2x_Iy_I)\xi\eta+$$
$$+\,(1-2z_I-2x_Iy_I)\Big[(1-2x_I)^2(y-z)^2h^2+$$
$$+\,2(1-2x_I)(y_I-z_I)(1-2z_I-2x_Iy_I)h + 4x_I^2y_I^2\Big]\Big\}.$$

Substituting (6) in ϑ_x and ϑ_y we obtain:

$$x_W = \frac{(1-2x_I)h^2 - 2(1-x_I)h + 2}{2},$$
$$y_W = \frac{h\big[4x_I^2y_I - h(1-2x_I)(1-2z_I-2x_Iy_I)\big]}{4x_I^2}.$$

In addition using the equality $z_W = 1 - x_W - y_W$ we find for z_W that:

$$z_W = \frac{h\Big[4x_I^2z_I - h(1-2x_I)(1-2y_I-2z_Ix_I)\Big]}{4x_I^2}.$$

If the equation of ω_0 is $\omega_0 = b_{23}yz + b_{31}zx + b_{12}xy + b_{11}x + b_{22}y + b_{33}z = 0$, then for the coordinates of the points X, Y, Z and the symmetric points X and Y with respect to W, we obtain the equalities:

$$b_{12} = 4x_I^2z_I^2, \quad b_{23} = 4x_I^4, \quad b_{31} = 4x_I^2y_I^2,$$
$$b_{11} = h^2(1-2x_I)^2(y_I-z_I)^2, \quad b_{22} = x_I^2\Big[h(1-2x_I) - 2z_I\Big],$$
$$b_{33} = x_I^2\Big[h(1-2x_I) - 2y_I\Big].$$

Solving the system with the equations of $\bar{k}(O)$ and ω_0 together with $x+y+z=1$, we deduce that the two curves have only one common point $T(x_T,\, y_T,\, z_T)$ with coordinates:

$$x_T = \frac{x_I^2\Big[h(1-2x_I) - 2y_I\Big]\Big[h(1-2x_I) - 2z_I\Big]}{\tau},$$
$$y_T = \frac{hy_I(z_I-y_I)(1-2x_I)\Big[h(1-2x_I) - 2y_I\Big]}{\tau},$$
$$z_T = \frac{hz_I(y_I-z_I)(1-2x_I)\Big[h(1-2x_I) - 2z_I\Big]}{\tau},$$

where

$$\tau = (1-2x_I)^2(1-2y_I)(1-2z_I)h^2 \\ - 2(1-x_I)(1-2x_I)(1-2y_I)(1-2z_I)h + 4x_I^2 y_I z_I.$$

Thus, Assertion 1 is proved.

Conclusion

Note that Assertion 1 is valid for central conics (Fig. 1, 2), but there exist Feuerbach's configurations consisted of parabolas. In such a case ω is a parabola with an infinite center $I \equiv I_0 \equiv O$ and it is homothetic to the parabola k with homothety center A. The curves $\bar{k}(O)$ and ω_0, which is a circumscribe one of $\triangle XYZ$, are parabolas with infinite point I. It is true the following:

Assertion 2. *The parabolas ω_0 and $\bar{k}(O)$ are tangent at T (Fig. 3).*

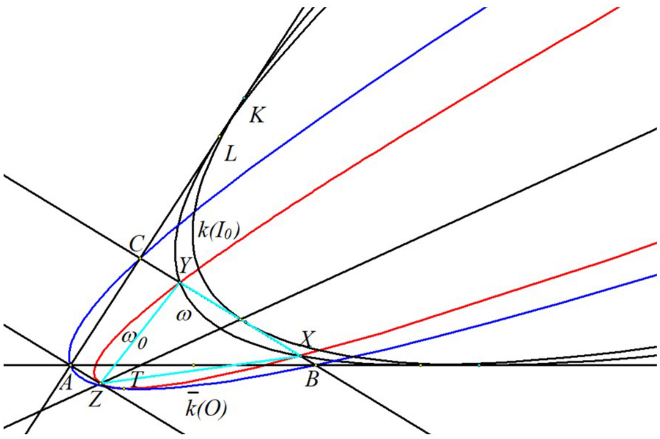

Figure 3.

The proof of Assertion 2 is similar to the proof of Assertion 1. The coordinates of the point T are expressed with the equalities:

$$x_T = \frac{(hx_I + y_I)(hx_I + z_I)}{y_I z_I (2h-1)^2},$$
$$y_T = \frac{h(z_I - y_I)(hx_I + y_I)}{z_I x_I (2h-1)^2},$$
$$z_T = \frac{h(y_I - z_I)(hx_I + z_I)}{x_I y_I (2h-1)^2}.$$

Since the homothety coefficient h could take different values, there exist an infinite set of tangent pairs of curves ω_0 and $\bar{k}(O)$for an arbitrary Feuerbach's configuration.

Finally, it should be noted that Assertions 1 and 2 are found using GSP, while the proofs are justified by the program Maple.

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