THE PROBLEM FOR ISOMORPHISM OF GROUP ALGEBRAS OF FINITE GROUPS OVER THE FIELD OF RATIONAL NUMBERS

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Abstract. *Let G and H be finite groups and $\mathbb{Q}$ be the field of rational numbers. The problem for isomorphism of group algebras over $\mathbb{Q}$ is formulated the following way: is it true that $\mathbb{Q}G \cong \mathbb{Q}H$ if and only if $G \cong H$? In this paper we prove that when $|G| < 27$, then $\mathbb{Q}G \cong \mathbb{Q}H$ always implies $G \cong H$. Furthermore, we construct an example that shows that when $|G| = 27$ then $\mathbb{Q}G \cong \mathbb{Q}H$ does not imply $G \cong H$.*

Key words: Semisimple Group Algebra; Isomorphism of Group Algebras; Commutant; Wedderburn Decomposition.

1. Introduction

Let G and H be finite groups, $\mathbb{Q}$ be the field of rational numbers and the group algebras $\mathbb{Q}G$ and $\mathbb{Q}H$ be isomorphic over $\mathbb{Q}$. If G and H are abelian groups, then from the Theorem of Perlis – Walker [1, Theorem 3] follows that G and H are isomorphic. This result also holds for some classes of non-abelian groups. We will show that if $|G| < 27$, then $\mathbb{Q}G \cong \mathbb{Q}H$ implies $G \cong H$ for arbitrary finite groups G and H. Additionally we construct an example that shows that when $|G| = 27$ then it is possible $\mathbb{Q}G \cong \mathbb{Q}H$ does not imply $G \cong H$.

2. Initial results

Lemma 2.1. *Let G and H are groups, $|G| = 8$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: There are only two non-isomorphic non-abelian groups of order 8 – the quaternion group and the dihedral group. In the decomposition of the rational group algebra of the quaternion group there is a simple component, isomorphic to division ring of the quaternions over $\mathbb{Q}$. But in the decomposition of the algebra of the dihedral group there is no such component and there is a matrix algebra with dimension 2^2 over $\mathbb{Q}$. □

Lemma 2.2. *Let G and H are groups, $|G| = 12$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: There are three non-isomorphic non-abelian groups of order 12. They are with defining relations, respectively 1) $a^4 = 1$, $b^3 = 1$, $a^{-1}ba = b^2$; 2) $a^2 = 1$, $b^6 = 1$, $a^{-1}ba = b^5$; 3) $a^3 = 1$, $b^2 = 1$, $c^2 = 1$, $a^{-1}ba = c$, $a^{-1}ca = bc$, $bc = cb$ and this group is isomorphic to the alternating group of degree 4. The quotient groups relative to the commutant of those groups are respectively: 1) a cyclic group of order 4; 2) the Klein four-group; 3) a cyclic group of order 3. All those groups are non-isomorphic and so the Lemma is proved. □

Lemma 2.3. *Let G and H are groups, $|G| = 16$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: There are nine non-isomorphic non-abelian groups of order 16. They are with defining relations, respectively: 1) $a^2 = 1$, $b^2 = 1$, $c^4 = 1$, $a^{-1}ba = bc^2$, $ac = ca$, $bc = cb$; 2) $a^2 = 1$, $b^4 = 1$, $c^2 = 1$, $a^{-1}ba = bc$, $ac = ca$, $bc = cb$; 3) $a^4 = 1$, $b^4 = 1$, $a^{-1}ba = b^3$; 4) $a^2 = 1$, $b^8 = 1$, $a^{-1}ba = b^5$; 5) $a^2 = 1$, $b^2 = c^2$, $c^4 = 1$, $b^{-1}cb = c^3$, $ab = ba$, $ac = ca$; 6) $a^2 = 1$, $b^2 = 1$, $c^4 = 1$, $b^{-1}cb = c^3$, $ab = ba$, $ac = ca$; 7) $a^2 = 1$, $b^8 = 1$, $a^{-1}ba = b^3$; 8) $a^2 = 1$, $b^8 = 1$, $a^{-1}ba = b^7$; 9) $a^2 = b^4$, $b^8 = 1$, $a^{-1}ba = b^7$ and this group is isomorphic to the quaternion group of order 16.

The quotient group relative to the commutant of the groups 1), 5) and 6) is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In the Wedderburn decomposition of the group algebra in case 1) there is a simple component $M_2(\mathbb{Q}(\varepsilon_4))$, where $\varepsilon_4^2 = -1$, but in the decompositions of the algebras in cases 5) and 6) there is no such component. In the Wedderburn decompositions of the group algebra in case 5) there is a simple component $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ (we use the notation from [2, 1.6]), isomorphic to the division ring of the quaternions over $\mathbb{Q}$, but in the decomposition of the algebra in case 6) there is no such component. Therefore those three algebras are non-isomorphic.

The quotient group relative to the commutant of the groups 2), 3) and 4) is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_4$. In the Wedderburn decomposition of the group algebra in case 2) there is a simple component $M_2(\mathbb{Q})$. In the Wedderburn decomposition of the group algebra i case 3) there are simple components $M_2(\mathbb{Q})$ and $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. In the Wedderburn decomposition of the group algebra in case 4) there is a simple component

$M_2(\mathbb{Q}(\varepsilon_4))$, where $\varepsilon_4^2 = -1$. Therefore those three algebras are not isomorphic. We note that the noncommutative simple components in cases 1) and 4) are isomorphic, but their commutative parts are not isomorphic.

The quotient group relative to the commutant of the groups 7), 8) and 9) are isomorphic to the Klien four-group. In the Wedderburn decomposition of the group algebra in case 7) there is a simple component $M_2(\mathbb{Q}(\varepsilon))$, where $\varepsilon^2 = -2$, but in cases 8) and 9) there is no such component. In the Wedderburn decomposition of the group algebra in case 9) there is a simple component $\left(\frac{-1,-2}{Q(\varepsilon)}\right)$, where $\varepsilon^2 = 2$, but in case 8) there is no such component. Therefore those three algebras are not isomorphic.

So the Lemma is proved. □

Lemma 2.4. *Let G and H are groups, $|G| = 18$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: There are three non-isomorphic non-abelian groups of order 18. They are with defining relations, respectively: 1) $a^2 = 1$, $b^9 = 1$, $a^{-1}ba = b^8$; 2) $a^6 = 1$, $b^3 = 1$, $a^{-1}ba = b^2$; 3) $a^2 = 1$, $b^3 = 1$, $c^3 = 1$, $a^{-1}ba = b^2$, $a^{-1}ca = c^2$, $bc = cb$. The quotient group relative to the commutant of group 2) is a cyclic group of order 6, and for the groups 1) and 3) it is a cyclic group of order 2. The Wedderburn decomposition of the group algebra in case 1) is $\mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}(\varepsilon + \varepsilon^8))$, where ε is a primitive 9-th root of 1 and $\varepsilon^6 + \varepsilon^3 + 1 = 0$, and the Wedderburn decomposition of the group algebra in case 3) is $\mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q})$.

The three algebras are non-isomorphic and therefore the Lemma is proved. □

Lemma 2.5. *Let G and H are groups, $|G| = 20$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: There are three non-isomorphic non-abelian groups of order 20. They are with defining relations, respectively: 1) $a^4 = 1$, $b^5 = 1$, $a^{-1}ba = b^2$; 2) $a^4 = 1$, $b^5 = 1$, $a^{-1}ba = b^4$; 3) $a^2 = 1$, $b^{10} = 1$, $a^{-1}ba = b^9$. The quotient group relative to the commutant of group 3) is of order 2, and for the groups 1) and 2) is of order 4. In the Wedderburn decomposition of the group algebra in case 1) there is a simple component $M_4(\mathbb{Q})$, but in case 2) there is no such component. The three algebras are non-isomorphic and the Lemma is proved. □

Lemma 2.6. *Let G and H are groups, $|G| = 24$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: There are twelve non-isomorphic, non-abelian groups of order 24. They are with defining relations, respectively: 1) $a^2 = b^6$, $b^{12} = 1$, $a^{-1}ba = b^7$; 2) $a^2 = 1$, $b^{12} = 1$, $a^{-1}ba = b^7$; 3) $a^8 = 1$, $b^3 = 1$, $a^{-1}ba = b^2$; 4) $a^4 = 1$, $b^6 = 1$, $a^{-1}ba = b^5$; 5) $a^2 = 1$, $b^{12} = 1$, $a^{-1}ba = b^5$; 6) $a^2 = 1$, $b^2 = 1$, $c^6 = 1$, $b^{-1}cb = c^5$, $ab = ba$, $ac = ca$; 7) $a^2 = b^2$, $b^4 = 1$, $c^3 = 1$, $a^{-1}ba = b^3$, $b^{-1}cb = c^2$, $ac = ca$; 8) $a^2 = 1$, $b^4 = 1$, $c^3 = 1$, $a^{-1}ba = b^3$, $b^{-1}cb = c^2$, $ac = ca$; 9) $a^2 = 1$, $b^4 = 1$, $c^3 = 1$, $a^{-1}ba = b^3$, $a^{-1}ca = c^2$, $bc = cb$; 10) $a^6 = 1$, $b^2 = 1$, $c^2 = 1$, $a^{-1}ba = c$, $a^{-1}ca = bc$, $bc = cb$; 11) $a^3 = 1$, $b^2 = c^2$, $c^4 = 1$, $a^{-1}ba = c$, $a^{-1}ca = bc$, $b^{-1}cb = c^3$; 12) $a^2 = 1$, $b^3 = 1$, $c^2 = 1$, $d^2 = 1$, $a^{-1}ba = b^2$, $a^{-1}da = cd$, $b^{-1}cb = cd$, $b^{-1}db = c$, $ac = ca$, $dc = dc$ and this group is isomorphic to the symmetric group of degree 4.

The quotient group relative to the commutant of the groups 1) and 2) is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_6$, the quotient group of the groups 4) and 5) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, and the quotient group of the groups 7), 8) and 9) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. For the rest of the groups the quotient groups relative to the commutant are not isomorphic. The Wedderburn decomposition of the group algebra in case 1) has a simple component $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, but in the decomposition of the group algebra in case 2) there is no such component. The Wedderburn decomposition of the group algebra in case 4) has a simple component $\left(\frac{-1,-3}{\mathbb{Q}}\right)$, but in the decomposition of the group algebra in case 5) there is no such component. The Wedderburn decomposition of the group algebra in case 7) has a simple component $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, but in the decomposition of the group algebra in cases 8) and 9) there is no such component. The Wedderburn decomposition of the group algebra in case 8) has a simple component $M_2(\mathbb{Q}(\varepsilon))$, where $\varepsilon^2 = -3$, but in the decomposition of the group algebra in case 9) there is no such component. The Wedderburn decomposition of the group algebra in case 9) has a simple component $M_2(\mathbb{Q}(\varepsilon))$, where $\varepsilon^2 = 3$, but in the decomposition of the group algebra in case 8) there is no such component. So the Lemma is proved. □

3. Main result

Theorem 3.1. *Let G and H are groups, $|G| < 27$ and $\mathbb{Q}$ be the field of rational numbers. If $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras, then $G \cong H$.*

Proof: We note that the conditions $|G| = |H|$ and $G/G^{(1)} \cong H/H^{(1)}$ immediately follow from the isomorphism $\mathbb{Q}G \cong \mathbb{Q}H$. From them it follows that if G is an abelian group, then the theorem holds. Then the remaining cases to examine are those, in which G is non-abelian group of order 8, 12, 16, 18, 20 or 24. But considering Lemmas 1 to 6, the Theorem is proved in full. □

4. An example of breach of the Problem for isomorphism when $|G| = 27$

Let p be an odd prime and G and H be groups of order p^3, defined with generators and defining relations in the following way:

$$G = \left\{a,\, b \,|\, a^p = 1,\ b^{p^2} = 1,\ a^{-1}ba = b^{p+1}\right\},$$
$$H = \left\{x,\, y,\, z \,|\, x^p = 1,\ y^p = 1,\ z^p = 1,\ x^{-1}yx = yz,\ xz = zx,\ yz = zy\right\}.$$

The groups G and H are not isomorphic because in G there is an element of order p^2, while in H there is no such element.

The minimal central idempotents of the group algebra $\mathbb{Q}G$ are:

$$e_1 = \frac{1}{p^3}\sum_{\lambda\in G}\lambda, \quad e_2 = \frac{1}{p^3}\left(p - \sum_{j=0}^{p-1} a^j\right)\left(\sum_{j=0}^{p-1} b^j\right)\left(\sum_{j=0}^{p-1} b^{pj}\right),$$
$$e_{1i} = \frac{1}{p^3}\left(\sum_{j=0}^{p-1} a^j b^{ip}\right)\left(p - \sum_{j=0}^{p-1} b^j\right)\left(\sum_{j=0}^{p-1} b^{pj}\right) \quad \text{for} \quad i = 1,\, 2,\, \ldots,\, p,$$
$$e = 1 - \frac{1}{p}\sum_{j=0}^{p-1} b^{pj}.$$

Their corresponding minimal components are: $\mathbb{Q}Ge_1 \cong \mathbb{Q}$, $\mathbb{Q}Ge_2 \cong \mathbb{Q}(\varepsilon_p)$, where ε_p is a primitive p-th root of 1, $\mathbb{Q}Ge_{1i} \cong \mathbb{Q}(\varepsilon_p)$, $\mathbb{Q}Ge \cong M_p(\mathbb{Q}(\varepsilon_p))$. Therefore the Wedderburn decomposition of $\mathbb{Q}G$ is:

$$\mathbb{Q}G \cong \mathbb{Q} \oplus (p+1)\mathbb{Q}(\varepsilon_p) \oplus M_p(\mathbb{Q}(\varepsilon_p)).$$

The minimal central idempotents of the group algebra $\mathbb{Q}H$ are of the same form as of $\mathbb{Q}G$ where a is replaced by x, b is replaced by y and b^p is replaced by z. The isomorphisms of the corresponding minimal components of $\mathbb{Q}H$ are also true as for $\mathbb{Q}G$. From here it follows that the Wedderburn decomposition of $\mathbb{Q}H$ is the same as of $\mathbb{Q}G$, i.e. $\mathbb{Q}G \cong \mathbb{Q}H$.

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