SOME RELATIONS BETWEEN THE SHAPE CURVATURES OF A THREE-DIMENSIONAL FRENET CURVE AND ITS ASSOCIATED FOUR-DIMENSIONAL FRENET CURVE

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Abstract. *In this research, we discuss differential-geometric invariants of a Frenet curve with respect to the group of direct similarities in a four-dimensional Euclidean space. In terms of the arc-length parameter, the relations between the shape curvatures of a three-dimensional Frenet curve and the shape curvatures of its associated four-dimensional Frenet curve are obtained.*

Key words: Frenet curves, Non-helical curves, General helices, Circular helices, Shape curvatures.

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1. Introduction

The Euclidean curvatures are well-known differential-geometric invariants that define a curve up to a rigid motion in n-dimensional Euclidean space (Euclidean n-space) $\mathbb{E}^n$, $n \in \mathbb{N}$, $n \geqq 2$. Our investigation are restricted to regular curves of order n in $\mathbb{E}^n$. In other words, we assume that every curve $\boldsymbol{\alpha} : I \subset \mathbb{R} \longrightarrow \mathbb{E}^n$, possesses derivatives up to order n and for any $t \in I$ the derivatives $\boldsymbol{\alpha}'(t)$, $\boldsymbol{\alpha}''(t)$ ... $\boldsymbol{\alpha}^{(n)}(t)$ are linear independent vectors in the n-dimensional real vector space $\mathbb{R}^n$. Those curves are named Frenet curves. Moreover, the Frenet curve can be determined up to a direct similarity of $\mathbb{E}^n$ by $n-1$ functions called shape curvatures. They are introduced in [3], [4] and [5]. In the presented paper, we construct a new curve in $\mathbb{E}^4$ that is associative to a given space curve in $\mathbb{E}^3$. Three classes of curves are considered, which are circular helices, general helices and non-helical curves. The relations between the differential-geometric invariants of the corresponding curves make it possible to study the properties of one curve through the other and vice versa.

2. Frenet curves in the Euclidean three-space

The Euclidean 3-space $\mathbb{E}^3$ is regarded as an affine space, and it has a vector space $\mathbb{R}^3$ that is related to it. The position column vector from

$\mathbb{R}^3$ can be used to represent any point in $\mathbb{E}^3$. The scalar (or dot) product $\mathbf{a}\cdot\mathbf{b} \in \mathbb{R}$ and the vector cross product $\mathbf{a}\times\mathbf{b} \in \mathbb{R}^3$ are well-known operations for any two column vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

Assume $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ be a curve described by a vector parametric equation

$$\boldsymbol{\alpha}(t) = (x(t),\, y(t),\, z(t))^T, \quad t \in I. \tag{1}$$

on an interval $I \subseteq \mathbb{R}$. The coordinate functions $x(t)$, $y(t)$, $z(t)$ are supposed to have continuous derivatives up to order 3. For a Frenet curve $\boldsymbol{\alpha}$ the Euclidean curvatures of $\boldsymbol{\alpha}$ in $\mathbb{E}^3$ (a curvature $\varkappa$ and a torsion τ) are determined by

$$\varkappa(t) = \frac{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|}{\|\boldsymbol{\alpha}'(t)\|^3} > 0,\ \tau(t) = \frac{\big(\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\big) \cdot \boldsymbol{\alpha}'''(t)}{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|^2} \neq 0 \tag{2}$$

Moreover, there are three unit vectors

$$\begin{gathered} \mathbf{t}(t) = \frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}, \quad \mathbf{n}(t) = \frac{\big(\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\big) \times \boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|.\|\boldsymbol{\alpha}'(t)\|}, \\ \mathbf{b}(t) = \frac{\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)}{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|} \end{gathered} \tag{3}$$

defined at any point $\boldsymbol{\alpha}(t)$ of the curve. A positively orientated orthonormal basis known as a Frenet frame is formed by these three vectors.

In the case that the Frenet curve $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ is a unit-speed curve, namely, $\|\boldsymbol{\alpha}'(t)\| = 1$ for any $t \in I$, the parameter "t" is usually substituted with the parameter "s", and the vector equation (1) is referred to as an arc-length parametrisation of $\boldsymbol{\alpha}$. In this instance, a more straightforward form of the formulas (2) and (3) can be used. Especially, the torsion and curvature are

$$\varkappa(s) = \|\boldsymbol{\alpha}''(s)\| > 0 \quad \text{and} \quad \tau(s) = \frac{\big(\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)\big) \cdot \boldsymbol{\alpha}'''(s)}{\|\boldsymbol{\alpha}''(t)\|^2} \neq 0, \tag{4}$$

respectively.

The Frenet curves in $\mathbb{E}^3$ belong to three significant classes: circular helices as a class and their extension general helices and non-helical curves, respectively. The arc-length parametrisation of any circular helix is as follows:

$$\boldsymbol{\alpha}(s) = \big(p\cos(as), p\sin(as), bs\big)^T,\ a \neq 0, b \neq 0, p \neq 0,\ a^2p^2 + b^2 = 1. \tag{5}$$

It has a constant curvature and torsion. An alternative parametrization of an arbitrary unit-speed circular helix is

$$\boldsymbol{\alpha}(s) = \left(\sqrt{\frac{1-b^2}{a^2}}\cos(as), \sqrt{\frac{1-b^2}{a^2}}\sin(as), bs\right)^T, \ a,\, b \neq 0, b \in (-1,1).$$

The curvature and the torsion of such a curve are $\varkappa(s) = \sqrt{a^2\left(1-b^2\right)}$ and $\tau(s) = ab$. Any curve in $\mathbb{E}^3$ whose tangent vectors form a constant angle with a fixed unit vector is referred to as a general helix or curve of constant slope. This fixed vector can be thought of as a vector parallel to the z-axis without losing generality. Then,

$$\boldsymbol{\alpha}(s) = \big(x(s),\, y(s),\, bs\big)^T,\ s \in I \subseteq \mathbb{R},\ b \in \{(-1,0) \cup (0,1)\} \tag{6}$$

and $(x'(s))^2 + (y'(s))^2 + b^2 = 1$ for any $s \in I$. A Frenet curve in $\mathbf{E}^3$ is a general helix if and only if the ratio of its curvature to torsion is constant (see Lemas 8.18 and 8.19 in [9]). Naturally, the class of general helices includes all circular helices as a subclass. Izumiya and Takeuchi have determined relations between plane curves and general helices in [11]. Ali (see [1]) studies parametric representations of general helices with a given curvature function and a given constant angle between the tangent vectors and a fixed unit vector. Examples of general helices with a unit speed parametrization can be found in [6] and [7]. The differential geometry of space curves is described in more detail in [2] and [9].

3. Frenet curves in the Euclidean four-space

We consider the Euclidean four-space $\mathbb{E}^4$ to be an affine space with column four-dimensional vectors in its corresponding real vector space $\mathbb{R}^4$. Accordingly, the position vector $\boldsymbol{X} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ may be used to identify any point $X \in \mathbb{E}^4$. There is a standard scalar (or dot) product of two vectors in the vector space $\mathbb{R}^4$. If $\mathbf{U} = (u_1, u_2, u_3, u_4)^T$ and $\mathbf{V} = (v_1, v_2, v_3, v_4)^T$ are four-dimensional vectors, then the scalar product of $\mathbf{U}$ and $\mathbf{V}$ is the real number $\mathbf{U}\cdot\mathbf{V} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ and the norm of the vector $\mathbf{U}$ is $\|\mathbf{U}\| = \sqrt{\mathbf{U}\cdot\mathbf{U}} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$.

Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{E}^4$ be a unit-speed curve of class C^4 defined on an interval $I \subseteq \mathbb{R}$ by a vector parametric equation

$$\boldsymbol{\gamma}(s) = \big(\gamma_1(s),\, \gamma_2(s),\, \gamma_3(s), \gamma_4(s)\big)^T, \quad s \in I. \tag{7}$$

This suggests that $\boldsymbol{\gamma}'(s) = \frac{d}{ds}\boldsymbol{\gamma}(s)$ is a unit vector for each $s \in I$ and that the coordinate functions $\gamma_i(s)$, $i = 1, 2, 3, 4$ have continuous derivatives up to order 4.

The curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{E}^4$ given by (7) is a unit-speed Frenet curve if the vectors $\boldsymbol{\gamma}'(s)$, $\boldsymbol{\gamma}''(s) = \frac{d}{ds}\boldsymbol{\gamma}'(s)$, $\boldsymbol{\gamma}'''(s) = \frac{d}{ds}\boldsymbol{\gamma}''(s)$ and $\boldsymbol{\gamma}^{(4)}(s) = \frac{d}{ds}\boldsymbol{\gamma}'''(s)$ are linearly independent for any $s \in I$, or equivalently $det\left(\boldsymbol{\gamma}^{'}(s), \boldsymbol{\gamma}^{''}(s), \boldsymbol{\gamma}^{'''}(s),\right.$ $\left.\boldsymbol{\gamma}^{(4)}(s)\right) \neq 0$ for any $s \in I$. According to [10], we can examine four unit vector functions $\mathbf{T}(s) = \boldsymbol{\gamma}'(s)$, $\mathbf{N}_1(s)$, $\mathbf{N}_2(s)$, $\mathbf{N}_3(s)$ and three real-valued curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ satisfying the following conditions for any value of the parameter s:

1. the vectors $\mathbf{T}(s)$, $\mathbf{N}_1(s)$, $\mathbf{N}_2(s)$, $\mathbf{N}_3(s)$ form a positively oriented, ordered basis of $\mathbb{R}^4$;
2. the Frenet-Serret equations

$$\begin{aligned} &\mathbf{T}'(s) = \kappa_1(s)\mathbf{N}_1(s),\ \mathbf{N}_1'(s) = -\kappa_1(s)\mathbf{T}(s) + \kappa_2(s)\mathbf{N}_2(s) \\ &\mathbf{N}_2'(s) = -\kappa_2(s)\mathbf{N}_1(s) + \kappa_3(s)\mathbf{N}_3(s),\ \mathbf{N}_3'(s) = \kappa_3(s)\mathbf{N}_2(s), \end{aligned} \tag{8}$$

as they are known, hold.

Banchoff and Lowett provided explicit curvature formulas and suggested an additional recursive method for figuring out the Frenet frame in [2]. The unit-speed Frenet curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{E}^4$ is assumed to be parameterized by (7). Proposition 3.1.9 in Banchoff and Lovett's book [2] shows that the curvatures of the curve $\boldsymbol{\gamma}$ in $\mathbb{E}^4$ can be expressed explicitly by the matrices $B_2(s) = \left(\boldsymbol{\gamma}'(s)\boldsymbol{\gamma}''(s)\right)$, $B_3(s) = \left(\boldsymbol{\gamma}'(s)\boldsymbol{\gamma}''(s)\boldsymbol{\gamma}'''(s)\right)$, $B_4(s) = \left(\boldsymbol{\gamma}'(s)\boldsymbol{\gamma}''(s)\boldsymbol{\gamma}'''(s)\boldsymbol{\gamma}^{(4)}(s)\right)$. More precisely, the first, the second and the third curvatures are given by

$$\kappa_1(s) = \sqrt{\det\left(B_2(s)^T B_2(s)\right)} = \|\boldsymbol{\gamma}''(s)\|, \tag{9}$$

$$\kappa_2(s) = \frac{1}{\left(\kappa_1(s)\right)^2}\sqrt{\det\left(B_3(s)^T B_3(s)\right)}, \tag{10}$$

$$\kappa_3(s) = \frac{1}{\left(\kappa_1(s)\right)^3\left(\kappa_2(s)\right)^2}\det\left(B_4(s)\right)). \tag{11}$$

Keep in mind that the requirement the curve $\boldsymbol{\gamma}$ to be a Frenet curve in $\mathbb{E}^4$ is equivalent to the condition $\boldsymbol{\gamma}$ to be a curve of class C^4 with curvatures $\kappa_1(s) > 0$, $\kappa_2(s) > 0$, and $\kappa_3(s) \neq 0$.

The unit-speed Frenet curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{E}^4$ defined by (7) has the unit tangent vector $\mathbf{T}(s) = \boldsymbol{\gamma}'(s)$ and the first, the second and the third unit normal vectors can be expressed as $\mathbf{N}_1(s) = \frac{1}{\|\gamma''(s)\|}\boldsymbol{\gamma}''(s)$, $\mathbf{N}_2(s) = \frac{1}{\kappa_2(s)}\left(\mathbf{N}_1'(s) + \kappa_1(s)\mathbf{T}(s)\right)$, $\mathbf{N}_3(s) = \frac{1}{\kappa_3(s)}\left(\mathbf{N}_2'(s) + \kappa_2(s)\mathbf{N}_1(s)\right)$.

4. Some relations between the shape curvatures of a three-dimensional Frenet curve and its associated four-dimensional Frenet curve

Let $I \subseteq \mathbb{R}$ be a zero-containing interval, and let $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ be a Frenet curve of class C^4 with an arc-length parametrisation and a parametrical equation

$$\boldsymbol{\alpha}(s) = \big(x(s),\, y(s),\, z(s)\big)^T, \quad s \in I. \tag{12}$$

According to the above assumptions, for any $s \in I$, the curvature $\varkappa(s)$ of $\boldsymbol{\alpha}$ is a positive real number, the torsion $\tau(s)$ of $\boldsymbol{\alpha}$ is a nonzero real number and the Frenet frame $\big(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\big)$ is well-defined. The unit-speed curves in $\mathbb{E}^4$ that are closely related to $\boldsymbol{\alpha}$ were studied in [8]. The following theorem provides the relations between the Euclidean curvatures of a Frenet curve in $\mathbb{E}^3$ and the Euclidean curvatures of its associated Frenet curve in $\mathbb{E}^4$.

Theorem 4.1. *[8] Let* (12) *be a parametrization of a unit-speed Frenet curve $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ of class C^4, and let $\varkappa(s)$ and $\tau(s)$ be the curvature and the torsion of $\boldsymbol{\alpha}$. Suppose that the curve $\boldsymbol{\beta}_1 : I \longrightarrow \mathbb{E}^4$ is defined by*

$$\boldsymbol{\beta}_1(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}x(s) \\ \frac{1}{\sqrt{2}}y(s) \\ \frac{1}{\sqrt{2}}z(s) \\ \frac{s}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\alpha}(s) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{s}{\sqrt{2}} \end{pmatrix}, \quad s \in I. \tag{13}$$

Then:
(i) $\boldsymbol{\beta}_1$ is also a unit-speed curve.
(ii) $\boldsymbol{\beta}_1$ is a unit-speed Frenet curve in $\mathbb{E}^4$ with curvatures

$$K_1(s) = \frac{\varkappa(s)}{\sqrt{2}}, \quad K_2(s) = \frac{\varkappa(s)}{\sqrt{2}}\sqrt{1 + 2\left(\frac{\tau(s)}{\varkappa(s)}\right)^2}, \qquad K_3(s) = -\frac{\sqrt{2}\left(\frac{\tau(s)}{\varkappa(s)}\right)'}{1 + 2\left(\frac{\tau(s)}{\varkappa(s)}\right)^2} \tag{14}$$

if and only if $\boldsymbol{\alpha}$ *is a non-helical curve, i.e.,* $\left(\frac{\tau(s)}{\varkappa(s)}\right)' \neq 0$ *for any* $s \in I$.

This result allows us to investigate relations between the shape curvatures of a three-dimensional Frenet curve $\boldsymbol{\alpha}$ and the corresponding four-dimensional Frenet curve $\boldsymbol{\gamma}$.

Theorem 4.2. *Let* (12) *be a parametrization of a unit-speed Frenet curve* $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ *of class* C^4*, and let* $\widetilde{\varkappa}(s) \neq 0$ *and* $\widetilde{\tau}(s)$ *be the shape curvature and the shape torsion of* $\boldsymbol{\alpha}$*, respectively. Suppose that the curve* $\boldsymbol{\gamma}_1 : I \longrightarrow \mathbb{E}^4$ *is defined by*

$$\boldsymbol{\gamma}_1(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}x(s) \\ \frac{1}{\sqrt{2}}y(s) \\ \frac{1}{\sqrt{2}}z(s) \\ \frac{s}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\alpha}(s) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{s}{\sqrt{2}} \end{pmatrix}, \quad s \in I. \tag{15}$$

Then $\boldsymbol{\gamma}_1$ *is a unit-speed Frenet curve in* $\mathbb{E}^4$ *with shape curvatures*

$$\widetilde{\kappa}_1(s) = \sqrt{2}\widetilde{\varkappa},\ \widetilde{\kappa}_2(s) = \sqrt{1+2\widetilde{\tau}^2} \ \text{and}\ \widetilde{\kappa}_3(s) = -\frac{2\widetilde{\tau}'}{\widetilde{\varkappa}\left(1+2\widetilde{\tau}^2\right)} \tag{16}$$

if and only if $\boldsymbol{\alpha}$ *is a non-helical curve, i.e.,* $\widetilde{\tau}(s) \neq const$ *for any* s.

Proof. The proof follows immediately from the equations (14) in Theorem 4.1 and the formulae $\widetilde{\kappa}_1(s) = \left(\frac{1}{\kappa_1(s)}\right)'$, $\widetilde{\kappa}_2(s) = \frac{\kappa_2(s)}{\kappa_1(s)}$ and $\widetilde{\kappa}_3(s) = \frac{\kappa_3(s)}{\kappa_1(s)}$ from [5]. □

Theorem 4.3. *Let* $\boldsymbol{\alpha} : I \rightarrow \mathbb{E}^3$ *be a unit-speed Frenet curve of class* C^4 *given by* (6)*. Then the curve* $\boldsymbol{\gamma}_2 : I \rightarrow \mathbb{E}^4$ *defined by*

$$\boldsymbol{\gamma}_2(s) = \left(x(s), y(s), \cos(bs), \sin(bs)\right)^T, \quad s \in I \tag{17}$$

is a unit-speed curve of class C^4 *with shape curvatures*

$$\widetilde{\kappa}_1(s) = \frac{\left(\int \widetilde{\varkappa} ds\right)^{-3} \widetilde{\varkappa}}{\sqrt{\left(\left(\int \widetilde{\varkappa} ds\right)^{-2} + b^4\right)^3}} \tag{18}$$

$$\widetilde{\kappa}_2(s) = \left(A^{-2}\left(\frac{\widetilde{\varkappa}^2+\widetilde{\tau}^2}{\left(\int\widetilde{\varkappa}ds\right)^4} + b^4\left(b^2-2A\right)\right) - \frac{\widetilde{\varkappa}^2A^{-3}}{\left(\int\widetilde{\varkappa}ds\right)^6}\right)^{\frac{1}{2}} \tag{19}$$

$$\begin{aligned}\widetilde{\kappa}_3(s) = {} & \frac{-b^3}{\left(\int\widetilde{\varkappa}ds\right)^3\widetilde{\tau}}\left(\widetilde{\tau}^2\left(\frac{3\widetilde{\varkappa}}{\left(\int\widetilde{\varkappa}ds\right)}+b\right)\right. \\ & \left. + b^3\left(1+\widetilde{\tau}^2-2\widetilde{\varkappa}^2+\int\widetilde{\varkappa}ds\left(\widetilde{\varkappa}'-b^2\int\widetilde{\varkappa}ds\right)\right)\right)\times \\ & \times\left(A\left(\frac{\widetilde{\varkappa}^2+\widetilde{\tau}^2}{\left(\int\widetilde{\varkappa}ds\right)^4}-2b^4+b^6\right)-\frac{\widetilde{\varkappa}^2}{\left(\int\widetilde{\varkappa}ds\right)^6}\right)^{-1},\end{aligned} \tag{20}$$

where $A = \left(\frac{1}{\left(\int\widetilde{\varkappa}ds\right)^2}+b^4\right)$.

Proof. It is clear that the vector function $\boldsymbol{\gamma}_2(s)$ has continuous derivatives

$$\begin{aligned}\boldsymbol{\gamma}_2'(s) &= \left(x'(s), y'(s), -b\sin(bs), b\cos(bs)\right)^T \\ \boldsymbol{\gamma}_2''(s) &= \left(x''(s), y''(s), -b^2\cos(bs), -b^2\sin(bs)\right)^T \\ \boldsymbol{\gamma}_2'''(s) &= \left(x'''(s), y'''(s), b^3\sin(bs), -b^3\cos(bs)\right)^T \\ \boldsymbol{\gamma}_2^{(4)}(s) &= \left(x^{(4)}(s), y^{(4)}(s), b^4\cos(bs), b^4\sin(bs)\right)^T\end{aligned} \tag{21}$$

up to fourth order. The first derivative $\boldsymbol{\gamma}_2'(s)$ is a unit vector function for any s. Therefore, $\boldsymbol{\gamma}_2$ is a unit-speed curve. From (21) and [12] it follows that

$$\begin{aligned}&\det\left(\boldsymbol{\gamma}_2'(s), \boldsymbol{\gamma}_2''(s), \boldsymbol{\gamma}_2'''(s), \boldsymbol{\gamma}_2^{(4)}(s)\right) = \\ &= b^3\begin{vmatrix} x'''(s)+b^2x'(s) & y'''(s)+b^2y'(s) \\ x^{(4)}(s)+b^2x''(s) & y^{(4)}(s)+b^2y'' \end{vmatrix}.\end{aligned}$$

Replacing the determinant and derivatives' obtained expressions in (9), (10), and (11), as well as using the formulae $\widetilde{\kappa}_1(s) = \left(\frac{1}{\kappa_1(s)}\right)'$, $\widetilde{\kappa}_2(s) = \frac{\kappa_2(s)}{\kappa_1(s)}$ and $\widetilde{\kappa}_3(s) = \frac{\kappa_3(s)}{\kappa_1(s)}$ we obtain (18), (19), and (20). □

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