VOLATILITY STRUCTURE DURING FINANCIAL TURBULENCE

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Abstract. *The attempts to construct a model that correctly replicates the market realities reached the maturity to challenge the standard Brownian Motion (sBM) as the stochasticity driver of the Black-Scholes log-returns. Recent researches provide arguments to generalize the sBM with a fractional Brownian Motion (fBM) [9, 17]. The capability of fBM-based model to be in line with the contemporarily admitted stylized facts explains the quest for the relevant value of the Hurst parameter (H-index). The estimation of the H-index values requires data for the unobservable volatility. Our inputs for the calibration include high-frequency trades and quotes that are integrated (regularized) to get a reliable proxy for the instantaneous variance. The available data span across more than 20 years and cover a large part of the COVID19 period which allows us concluding whether the volatility during this turmoil was rough or smooth based on the value of the inferred H-index. Note that unlike the mathematical context where "smoothness" means differentiable function, here "smoothness" means to be smoother than the sBM; respectively, "roughness" means to be rougher than the sBM.*

Key words: volatility structure, fractional Brownian Motion, rough volatility, long memory, financial market misbehavior, Hurst parameter, stylized facts, high-frequency trading data.

Mathematics Subject Classification: 62P05, 60G22, 91G20.

JEL Classification: C02, D53, G01.

1. Introduction

The stylized facts play a key role in the way the humans comprehend the complicated and vibrating reality around them. The reality complexity is reduced to a degree that matches the current cognitive capacities. This simplification is needed and justifiable until it preserves some reality pivots called *stylized facts*. Actually, the stylized facts is what we decided *a priori* to find in the mirrored simplification of the reality. Hence, any bias in defining and enumerating the stylized facts results in a distorted image of the reality.

Some knowledge branches (like model validation) developed the epistemological concept of stylized facts, ways to establish new stylized facts, and thus secure dynamics of the stylized facts set [31]. Other areas (like the Financial Mathematics for example) do not have such toolkit; the academic community attempts to identify which features a financial model should not miss.

The establishment of stylized facts is important *per se* and also since it unlocks further activities, like developing financial models (see for example [23]), volatility forecasting [27], and portfolio optimization [22]. Financial models are developed to reduce the financial market complexity to a limited number (one or two but no more than that) of already established and agreed upon stylized facts; for example, a model is developed to reflect the jumps observed on the market, to capture the stochasticity of the volatility or both of them. Consequently, the model is rated on its capacity to capture this or that aspect of market activities.

The development of stylized facts got a substantial impetus in 2001 when Rama Cont [10] defined the stylized facts as 'A set of properties, common across many instruments, markets and time periods, has been observed by independent studies and classified as "stylized facts"' and solved the "model vs. data" dilemma in favor of the latter. Also, he introduced the challenges created by the availability of high-frequency data and underlined how important is to have processing power.

Cont explored the qualitative properties of the asset returns while the volatility remains subordinated to the returns in his research. Only four of totally eleven stylized facts deal with volatility features (Intermittency, Volatility clustering, Volume/volatility correlation, Asymmetry in time scales). A separate group of stylized facts comprises interactions between returns and volatility (Conditional heavy tails, Leverage effect).

Since 2001 the majority of academic efforts remains constrained in verifying that the above stylized facts are still valid. This approach underestimates an important market innovation: the volatility itself became an asset with the introduction of trading derivatives on the VIX Index (in 2004 for VIX futures and in 2006 for VIX options). The basic way to estimate the volatility via returns reinforces the importance of returns-related stylized facts.

Some recent researches establish volatility-related stylized facts when the volatility is estimated from returns. For example, Masset [30] identifies several volatility stylized facts (Horizontal dependence, Extreme events, leverage effects, Vertical dependence) for bull and bear market in emerging and mature markets. Ghosh and co-authors explore the stylized facts for crypto currencies [18].

Other studies establish stylized facts for the realized volatility or the historic volatility. For example, Baillie and co-authors focus on the long-memory of the realized volatility [1]. Zumbach [41] explored several stylized facts about the volatility and the volatility increments (Probability Density Function, Moments Scaling, Relative Excess Kurtosis, Lagged Correlation, Realized-Historical Volatility Correlation, Trend and Leverage Effects).

Also, some researchers check stylized facts of the implied volatility, like the mean-reversion property [21], the path-dependency [38], etc. which allows its prediction [20]. Sinclair [36] explores several stylized facts of the implied volatility – Volatility Level Dynamics, The Smile and its dynamics, Term Structure Dynamics.

It is emblematic to mention the numerous attempts to preserve the sBM as the pivotal tool to establish stylized facts: by end-2024 the book [26] got its 13th edition. It seems reasonable to ask to what extent our understanding and the admitted stylized facts keep pace with the market evolution. Back in the 80s financial markets were expected to match the Efficient Market Hypothesis [13]. Hence, random walk suffices to describe and forecast the market dynamics [26]. A couple of decades later the market ecosystem required the invention of the Fractal Market Hypothesis [32]. Stochastic tools more sophisticated than the random walk were necessary [24]. Hence, the sBM was replaced by the fBM even though the latter was added to the Financial Mathematics' toolkit by Mandelbrot and Van Ness back in 1968 [28]. Nowadays, there are attempts to apply the Adaptive Market Hypothesis to the current market realities [25]. But still there is no complete set of stylized facts to adequately describe what we observe on the markets.

Last but not least, let us mention that the available stylized facts have been designed to grasp the market in its equilibrium mode. The financial crisis are considered a non-representative exception that appears unexpectedly and market's memory absorbs its echoes. Key elements of the Fractals [29] and the Chaos Theory [33] were added to the financial toolkit but still no *vivant* set of turmoil stylized facts was nailed down from these theories.

The spillover of the COVID19 pandemic on financial markets exemplified the gaps in our understanding of the globalized markets in turbulence (especially the volatility). Consequently, attempts were done to close these gaps. Vera-Valdes [37] found that the pandemic resulted in a longer memory of the volatility for the VIX Index and the realized variance; also, some volatility measures got non-stationary. Bhattacharjee and co-authors [6] reported persistence and drop of the volatility for sectoral indexes on India's National Stock Exchange. Bentes [3] found increased volatility persistence in the G7 markets. Curto and Serrasqueiro [12] focused on three Cont's stylized facts (clustering, persistence and asymmetry) of the return volatility and observed differing reaction for eleven S&P500 sector subindices. Zhang and Fang [39] explored 5-min time series for CSI300 Index (China) and S&P500 (USA) and found that the multifractal characteristics increased during the pandemic period.

The purpose of this research is to derive the H-index time series for a vast data set, study its features during market equilibrium times and market unrests (especially during the COVID19 pandemic period), and check whether new volatility-related stylized facts might be established.

The paper is structured as follows. To achieve our goals, we consider in Section 1 the theoretical background for the volatility structure and the dynamics of the ruling paradigms. Next, we describe in Section 2 what algorithm we follow. In Section 3 we derive the H-index series and establish new stylized facts from their features. Finally, we draw conclusions and raise recommendations for further studies (Section 4).

2. Theoretical Background

Let us begin with an admitted stylized fact related to the at-the-money volatility skew (ATMVS). The ATMVS is described as follows:

$$\psi(\tau) := \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0}, \tag{1}$$

where σ_{BS} is the implied Black-Scholes volatility, k is the log-moneyness, and τ is time to expiry. Fig. 1 of [17] which exemplifies the ATMVS reads a curve that reaches infinity as time reaches zeros. On the volatility side, various approaches are developed to estimate the volatility both on the long-term end of the curve and on the short-term end; these methods differ in their capability to capture the above stylized fact.

On the returns side, models are developed with the purpose to adjust the basic Black-Scholes model to the contemporary realities and hence jump component is added to the models to reflect the infinity-reaching short-term end of the ATMVS curve (example for such models is the Merton jump diffusion model for the constant volatility and the Bates model for the stochastic volatility). Such model augmentation comes at the costs of calibration troubles – for example, the Bates model calibration requires that eight parameters are inferred.

Recent researches reveal that the ATMVS curve might be fitted by a power-law of certain parameters; for concrete fit parameters see [17]. This finding opens the door to generalize the sBM with the fBM. Adequately calibrated fBM-based model would pass the reality-checks without the burdensome jump component [17, 35].

The reliance on the fBM necessitates to define this process and briefly discuss its properties. The fBM B_t^H is a continuous and centered Gaussian process with covariance function (see [7], pp. 5–6):

$$E\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{2}$$

where H is the Hurst-index, $H \in (0,1)$, and $0 \leq s < t$.

The fBM is marked with several properties:

- $B_0^H = 0$ and $E\left[B_t^H\right] = 0$ for all $t \geq 0$.
- B_t^H has homogenous increments, i.e. $B_{t+s}^H - B_s^H$ has the same law as B_t^H.
- $E\left[\left(B_t^H\right)^2\right] = t^{2H}$.
- B_t^H is a self-similar process, i.e. for all $a > 0$ $\mathrm{Law}\big(B_{at}^H, t \geq 0\big) = \mathrm{Law}\big(a^H B_t^H, t \geq 0\big)$.
- it is not semi-martingale for $H \neq \frac{1}{2}$.
- it is not Markovian process for $H \neq \frac{1}{2}$.

The self-similarity property links two other fBM properties: the long-range dependence of increments and the Hölder continuity of any order less than the H-index, i.e. $E\left[|B_t^H - B_s^H|^{H-\epsilon}\right] \leq k|t-s|^{H-\epsilon}$ – see [19]. These two properties reflect the H-index value in the following way:

- when $H = \frac{1}{2}$, then we are talking of sBM;
- when $H \in \left(\frac{1}{2}, 1\right)$, then we have long-range dependence and smooth paths (smoother than the sBM);
- when $H \in \left(0, \frac{1}{2}\right)$, the process experiences counter-persistence and rough paths (rougher than the sBM).

3. Approach selected

The literature reveals several approaches to disentangle the long-range dependence from the Hölder continuity. First, focus on the long-memory property alone (Comte and Renault coined the Fractional Stochastic Volatility Model [9]); second, focus on the roughness only [17]; third, separate these two properties by applying Fractional Ornstein-Uhlenbeck Process, Cauchy Process or Brownian Semistationary Processes (see [2]).

fBM-based model application requires the estimation of the H-index value. A challenge for H-index calibration is the lack of a unified approach to follow [4, 5, 34]. Also, the spot volatility is a hidden market signal and only the realized variance is estimated.

The above challenges might be cured by leveraging the following approaches:

- run regression to exploit the monofractal property of the fBM (see [17] for details) which has been criticized by some authors for committing estimation error [11, 14]. Such implementation includes calculating moments of log-volatility differences:

$$m\left(q, \Delta\right) = \frac{1}{N}\sum_{k=1}^{N}\left|\ln\sigma_{k\Delta} - \ln\sigma_{(k-1)\Delta}\right|^{q}, \tag{3}$$

  where m is the moment of order q, Δ is the time lag (days), and σ is the volatility. Given the fBM monofractal property for various q, we anticipate to observe $m\left(q, \Delta\right) \propto \Delta_q^{\zeta}$. Hence, we can derive the H-index value by running regression $\zeta_q \approx q\hat{H}$.

- Run regression to exploit the auto-correlation function of the volatility (see [2] for details). Let us present the volatility as $y_t = \ln v_t = B_t^H$. Then, the variance is $var\,[y_t] = t^{2H}$ and the covariance is $cov\,[y_t, y_{t+1}] = \frac{1}{2}\{|t|^{2H} + |t+\Delta|^{2H} - \Delta^{2H}\}$. We get the following auto-correlation function

$$\rho\,(\Delta) = \frac{1}{2}\left\{1 + \left(1 + \frac{\Delta}{t}\right)^{2H} - \left(\frac{\Delta}{t}\right)^{2H}\right\}. \tag{4}$$

For sufficiently small $\frac{\Delta}{t}$, we get $\ln\left(1 - \rho\,(\Delta)\right) = a + 2H \ln \Delta$ which is the base for the regression.

4. Estimating H-index During the COVID19 Period

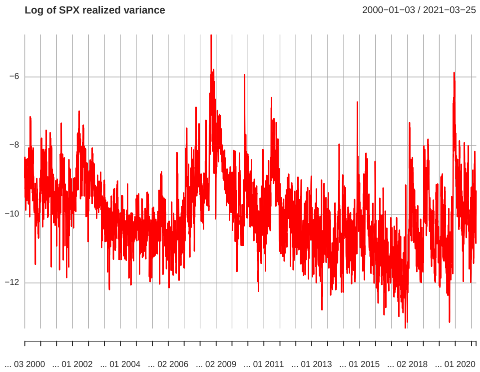
We model the volatility as $\sigma_t = c \exp\left(v B_t^H\right)$, where c, v are positive constants and B_t^H is a fBM.

Our research uses the data set from the online Oxford-Man library (`https://oxford-man.ox.ac.uk/research/realized-library/`). The set includes 31 indexes that cover the Americas (the USA, and a couple of Emerging Markets), Europe, and Asia (incl. some Emerging Markets). The time ranges from January 2000 to March 2021; in other words, data for the COVID19 pandemic period is available to us. We are talking of high-frequency series for 5-min realized variance (RV), 10-min RV, and various kernels (Tukey-Hanning, Two-Scale/Bartlett, and Non-Flat Parzen) are applied to integrate the RV. For more details about the features of the data see [16]. Figure 1.a exemplifies the data subset for S&P500 Index and 5-min integration of the RV via the Tukey-Hanning kernel.

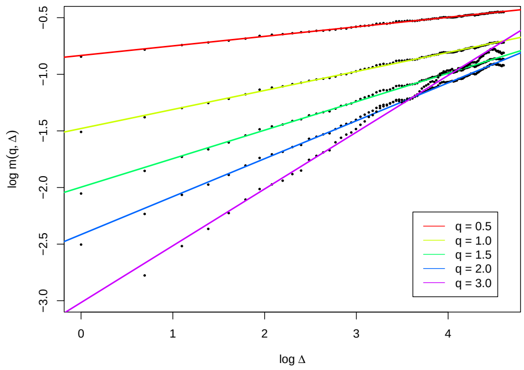
Exploring the data, we observe that data for the STI index is missing for the period 2008 – 2015 and for this reason the index is excluded from our research. The DJI data contain negative values for a couple of days. We removed from the data 10 dates with variance of 0. We believe the steps taken to secure the data quality cover a limited portion of the data and thus do not compromise our conclusions.

We challenged the approach defined by statement (3), confirmed the monofractal property $m\,(q, \Delta) \propto \Delta_q^{\zeta}$ which is compatible with [17] – see Figure 1.b. Also, we questioned the link between ζ_q and $\hat{H}$ for several indexes: S&P500 (Figure 2.a), STOXX50E (Figure 2.b), FTSE (Figure 2.c), and KSE (Figure 2.d). We find that the link between ζ_q and $\hat{H}$ for the latter 3 indexes is far from being linear. Table 1 reads the coefficients of a parabolic fit.

Figure 1. Realized Variance Series and Their Monofractal Property

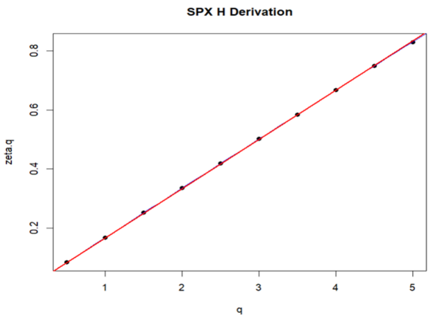


(a) Log 5-min Realized Variance for S&P500 Index with Tukey-Hanning Kernel

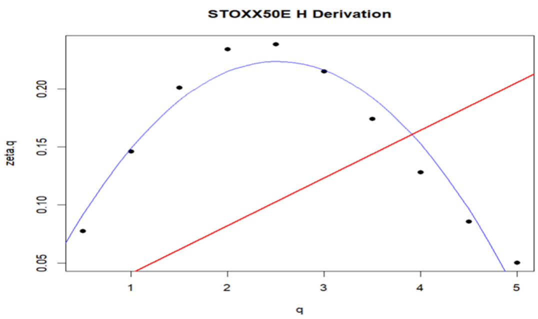


(b) Monofractal Property Check

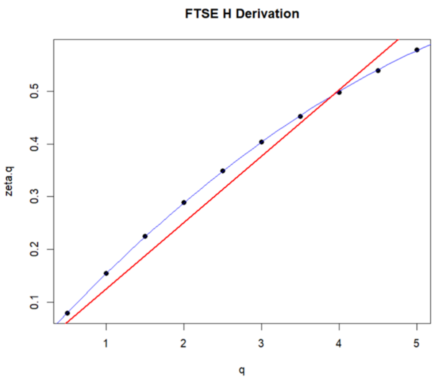
Figure 2. Derived H-index Values



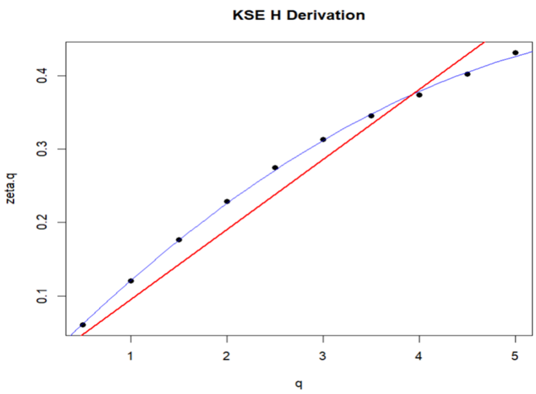
(a) H-index Derivation for S&P500 Index



(b) H-index Derivation for STOXX50E Index



(c) H-index Derivation for FTSE Index



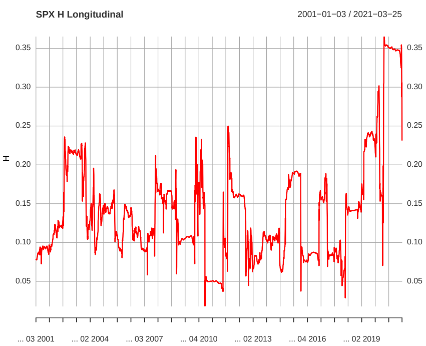
(d) H-index Derivation for KSE Index

Table 1. Quadratic Fit Coefficients

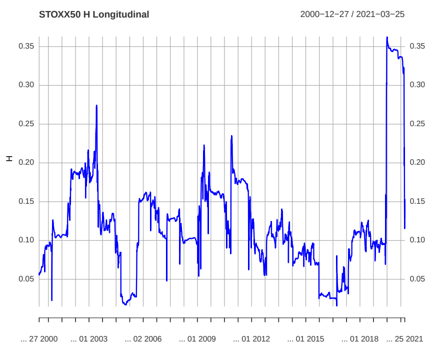
$FitCoefficient$	$S\&P500$	$STOXX50E$	KSE	$FTSE$
$intercept$	-0.0006898	0.01819	-0.002272	0.001306
$coef1$	0.1693142	0.1631	0.13306	0.162386
$coef2$	-0.0006323	-0.03238	-0.00948	-0.009459

Hence, we are not able to estimate the H-index by running $\zeta_q \approx q\hat{H}$ linear regression. This failure drove us to use the approach defined by statement (4). Figure 3.a reads the H-index series of S&P500. Next, we checked whether the same observations apply to other indexes; we find that the H-index series for $STOXX50E$ (for example) follow the same patterns (see Figure 3.b). We double check this result by drawing the two series S&P500 vs. STOXX50E (see Figure 3.c). We would like to exclude any role of the integrating kernel; hence, we run the same checks for the Parzen kernel (see Figure 3.d). Figure 3.e presents the H-index series for the COVID19 period only.

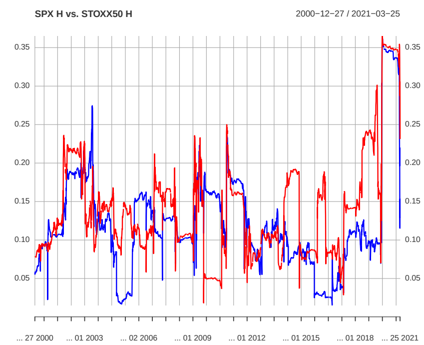
Figure 3. H-index Series



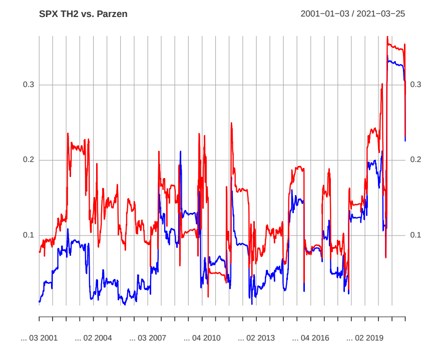
(a) H-index Series for S&P500 Index



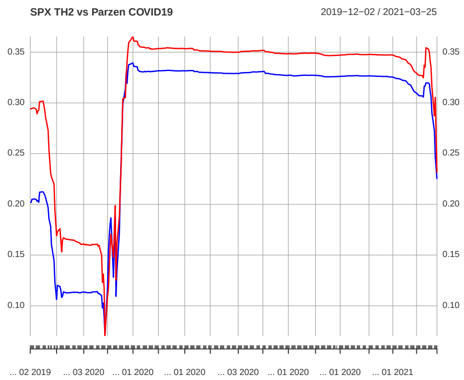
(b) H-index Series for STOXX50E Index



(c) Comparison H-index S&P500 vs. STOXX50E



(d) Comparison Tukey-Hanning vs. Parzen Kernel for S&P500 Index



(e) H-index Series During COVID19 Period.

5. Conclusions and further works

Our research was launched from the ubiquitous position where the dominant way to estimate volatility (as the standard deviation of asset log-returns) complicates the derivation of volatility-related stylized facts. There is no adopted algorithm to reveal volatility-related stylized facts nor short-list of them and the admitted volatility-related stylized facts lag behind the current Market Hypothesis. Another obstacle for us is the under-exploration of market turbulence; stylized facts of market crisis are mimicked as Market Inefficiency (i.e. a deviation from the Efficient Market Hypothesis), Market Failures, etc. and the financial crisis is not conceived as an intrinsic market status. Stylized facts for the historic/realized volatility are better researched than the features of the implied volatility.

Based on a representative data set for an extensive time period, we observe scaling property of the log-volatility for a large pool of market indexes. Also, we confirmed there is link $\zeta_q \propto \hat{H}$ but it is a non-linear. Our calculations read a material increase of the H-index values during the COVID19 period; we do not spot similar increase during other turmoils, like the Global Financial Crisis 2008/2009 for example. On Figure 3.e we distinguish several sub-periods of the COVID19 pandemic that match the timing split discovered by other authors (see for example [12, 3]).

We can summarize the key contribution of our research as a couple of new stylized volatility-related facts. We find that the log-volatility is rough since the Hurst-index value is $H < \frac{1}{2}$. Additionally, the H-index value varies within a range and moves in packages with transition period between the packages. We noted that the log-volatility becomes smoother during market misbehavior. We observe that the choice of the kernel type determines the H-index value but preserves the above conclusions.

We started our research from one stylized fact ($\psi(\tau) \to \infty$ as $\tau \to 0$) and established couple of stylized facts. We suggest further researches to create eco-system of the volatility-related stylized facts that would unleash new findings of academic and practical merits. Additional efforts are necessary to separate the volatility studies from the returns series and hence a new generation of volatility-estimation methods is needed; new ways of volatility estimation might inspire a new generation of rough volatility models in addition to the existing ones, like the rough-Heston, the extended rough-Heston, the rough-Bergomi model. The next explorations in this area of the Financial Mathematics might derive the H-index value by running non-linear regression $\zeta_q \approx q\hat{H}$ by leveraging the fit coefficients from Table 1.

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